Abundant New Exact Solutions of the Coupled Potential KdV Equation and the Modified KdV-Type Equation

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Exact solutions of nonlinear evolution equations (NLEEs) in soliton theory and their applications are studied. A powerful method is established to search for exact travelling wave solutions of NLEEs. We chose the coupled potential KdV equation and modified KdV-type equations presented by Foursov to illustrate the approach with the aid of Maple. As a result, eight families of exact solutions of the coupled potential KdV equation and nine families of exact solutions of the modified KdV-type equations are obtained, which contain new kink-like soliton solutions, kink-shaped solitons, bell-shaped solitons, periodic solutions, rational solutions and singular solitons. The properties of the solutions are shown in figures.

Key words: Coupled Potential KdV Equation, Modified KdV-type Equation; Soliton Solutions; Periodic Solutions; Rational Solutions.

1. Introduction

As is well known, searching for new exact solutions of new nonlinear evolution equations (NLEEs) is an important subject in soliton theory and its applications since the solitary wave phenomena observed by Scott Russell in 1834 [1]. Early in the study of soliton theory, the main attention was payed to the (1+1)-dimensional cases with few components, such as the KdV equation, Burgers equation, Boussinesq equation, etc. [1, 2]. But nonlinear evolution equations of many nonlinear phenomena contain more components and dimensions. Many nonlinear soliton equations in (2+1)-dimensional space, such as the generalized KdV equation and the Nizhnik-Novikov-Veselov equation were presented [3, 4]. Multi-component nonlinear evolution equations also were found [5].

More recently, by using the generalized symmetry method, Foursov [6] had derived eleven previously unknown classes of integrable equations. The integrable coupled potential KdV equation

$$u_t = u_{xxx} + 3uu_{xx} - 3vu_{xx} + 3u_x^2 + 3u^2u_x$$
$$- 6uvu_x + 3v^2u_x,$$
$$v_t = v_{xxx} - 3uv_{xx} + 3vv_{xx} + 3v_x^2 + 3u^2v_x$$
$$- 6uvv_x + 3v^2v_x. \tag{1}$$

and the modified KdV-type equations are

$$u_t = u_{xxx} + 3uu_{xx} - 3vu_{xx} + 3u_x^2 - 3u_xv_x + 3u^2u_x$$
$$- 24uvu_x + 3v^2u_x - 12u^2v_x + 6uvv_x,$$
$$v_t = v_{xxx} - 3uv_{xx} + 3vv_{xx} - 3u_xv_x + 3v_x^2 + 3u^2v_x$$
$$+ 6uvu_x - 12v^2u_x - 24uvv_x + 3v^2v_x. \tag{2}$$

When $v = 0$, (1) and (2) become the five-component equation

$$u_t = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x.$$

The exact travelling wave solutions of the systems (1) and (2) are still unknown. The aim of the present paper is to improve the extended tanh-function method [7, 8] and of our previous method [9 - 11], and to use it for exact solutions of (1) and (2) with the aid of Maple. That is to say, we solve the Riccati equation with a parameter to express solutions of the systems (1) and (2).

The extended tanh-function method [7, 8] is as follows:

For a given nonlinear partial differential equation (PDE), say in two variables, one has

$$F(u, u_t, u_x, u_{xx}...) = 0. \qquad (3)$$

If we consider a travelling wave solution of the form $u(x,t) = u(\xi), \xi = x - \lambda t + c$, then (3) reduces to an ordinary differential equation (ODE). The next step is to look for a formal solution

$$u(\xi) = \sum_{i=0}^{m} a_i w^i = a_0 + a_1 w + ... + a_m w^m, \quad (4)$$

where $w = w(\xi)$ satisfies the Riccati equation

$$\mathrm{d}\, w/\mathrm{d}\, \xi = b + w^2. \qquad (5)$$

where $\lambda, b, a_i (i = 0, 1, ..., m)$ are parameters. The parameter m can be determined by balancing the linear term of highest order with the nonlinear term in (3). Substituting (4) into (3) yields a set of algebraic equations of $\lambda, b, a_i (i = 0, 1, ..., m)$ because all coefficients of w^i have to vanish. From this set of equations, $\lambda, b, a_i (i = 0, 1, ..., m)$ can be obtained. In addition, we know that the general solution of (5) is

$$w_1 - \sqrt{-b}\tanh(\sqrt{-b}\xi),$$

$$w_2 = -\sqrt{-b}\coth(\sqrt{-b}\xi),\ b < 0;$$

$$w_1 = \sqrt{b}\tan(\sqrt{b}\xi),\ w_2 = -\sqrt{b}\cot(\sqrt{b}\xi),\ b > 0;$$

$$w = -\frac{1}{\xi},\ b = 0. \qquad (6)$$

The solutions obtained by using this method are of the form, for example, $a_0 + a_1 \tanh(k\xi) + a_2 \tanh^2(k\xi)$ +.... However a solution in the form $\alpha \operatorname{sech}(k\xi)$ of the mKdV equation is not obtained by using this method.

Recently we presented a method [9] in which only the second step is different from the above method, i. e., we searched for the solution

$$u(\xi) = \sum_{i=1}^{m} w^{i-1}[A_i w + B_i \sqrt{\mu_1(1 + \mu_2 w^2)}] + A_0, (7)$$

where $w = w(\xi)$ satisfies a Riccati equation

$$\mathrm{d}\, w/\mathrm{d}\, \xi = R(1 + \mu_2)w^2,\ \mu_1 = \pm 1, \mu_2 = \pm 1. \quad (8)$$

Equation (8) has the general solutions

$$w_1 = \tanh(R\xi),\ w_2 = \coth(R\xi),\ \mu_2 = -1,$$

$$w_1 = \tan(R\xi),\ w_2 = -\cot(R\xi),\ \mu_2 = 1.$$

Except for the rational solutions, the solutions obtained by our method contained all other ones obtained by the extended tanh-function method.

To overcome the disadvantage of the two methods, we would like to improve the ansatz (4) or (7) as follows:

$$u(\xi) = \sum_{i=1}^{m} w^{i-1}[A_i w + B_i \sqrt{1 + \operatorname{sgn}(b)w^2}]$$

$$+ A_0 + \sum_{j=0}^{n} D_j \xi^j, \qquad (9)$$

where $A_0, A_i, B_i (i = 1, ..., m), D_j (j = 0, 1, ..., n)$ are parameters to be determined. It is easy to see that when $B_i = D_j = 0$, the ansatz (9) just becomes Fan's ansatz (4). However if $B_i D_j \neq 0$, we may obtain new solutions that can not be found by using Fan's method [7, 8]. In what follows, we will apply our method to the systems (1) and (2), which will show that our method is more powerful to obtain more types of exact solutions containing soliton solutions.

2. Eight Families of Exact Solutions of System (1)

For the given system (1) we consider travelling wave solutions $u(x,t) = u(\xi), v(x,t) = v(\xi), \xi = x - \lambda t + c$. Then system (1) reduces to a set of ordinary differential equations

$$u''' + 3uu' - 3vu'' + 3u'^2 + 3u^2u' - 6uvu'$$

$$+ 3v^2u' + \lambda u' = 0,$$

$$v''' - 3uv'' + 3vv' + 3v'^2 + 3u^2v' - 6uvv'$$

$$+ 3v^2v' + \lambda v' = 0, \qquad (10)$$

where λ is a constant to be determined later and c is an arbitrary constant.

We suppose that the system (10) has solutions of the form

$$u = A_0 + A_1 w + D_1 \xi, \quad v = a_0 + a_1 w + d_1 \xi, \quad (11)$$

with $w = w(\xi)$ satisfying (5) and where A_0, A_1, D_1, a_0, a_1, d_2 are constants to be determined later.

With the aid of Maple, substituting (11) into (10) along with (5) and collecting all terms with the same power in $\xi^i w^j (i = 0, 1, 2; j = 0, 1, 2, 3, 4)$, yields a system of equations for $\xi^i w^j (i = 0, 1, 2; j = 0, 1, 2, 3, 4)$. Setting the coefficients of $\xi^i w^j (i = 0, 1, 2; j = 0, 1, 2, 3, 4)$ to zero yields the following set of over-determined algebraic polynomials with respect the unknowns λ, b, A_0, A_1, D_1, a_0, a_1, and d_1, namely

$$6A_1 + 9A_1^2 - 6a_1A_1 + 3A_1^3 - 6a_1A_1^2 + 3a_1^2A_1 = 0, \quad (12.1)$$

$$6A_1D_1 - 6d_1A_1 + 6A_1^2D_1 - 6A_1(A_1d_1 + a_1D_1)$$
$$+ 6A_1a_1d_1 = 0, \quad (12.2)$$

$$6A_0A_1 - 6a_0A_1 + 6A_0A_1^2 - 6A_1(A_0a_1 + a_0A_1)$$
$$+ 6a_0a_1A_1 = 0, \quad (12.3)$$

$$6A_0A_1D_1 - 6A_1(A_0d_1 + a_0D_1) + 6a_0d_1A_1 = 0, \quad (12.4)$$

$$8bA_1 + 12bA_1^2 - 6ba_1A_1 + 6A_1D_1 + 3A_0^1A_1$$
$$+ 3A_1^2(A_1b + D_1) - 6A_0a_0A_1 - 6a_1A_1(bA_1 + D_1)$$
$$+ 3a_0^2A_1 + a_1^2(bA_1 + D_1) + \lambda A_1 = 0, \quad (12.5)$$

$$6bA_1D_1 - 6bA_1d_1 + (bA_1 + D_1)$$
$$\cdot (3A_1D_1 + 3a_1d_1 - 6A_1d_1 - 6a_1D_1) = 0, \quad (12.6)$$

$$3D_1^2A_1 + 3d_1^2A_1 - 6d_1D_1A_1 = 0, \quad (12.7)$$

$$6bA_0A_1 - 6ba_0A_1 + (A_1b + D_1)$$
$$\cdot (6A_0A_1 + 6a_0a_1 - 6A_0a_1 - 6a_0A_1) = 0, \quad (12.8)$$

$$(3D_1^2 + 3d_1^2 - 6d_1D_1)(bA_1 + D_1) = 0, \quad (12.9)$$

$$(6A_0D_1 + 6a_0d_1 - 6A_0d_1 + a_0D_1)$$
$$\cdot (bA_1 + D_1) = 0, \quad (12.10)$$

$$2b^2A_1 + 3(b^2A_1^2 + D_1^2 + 2bA_1D_1)$$
$$+ (bA_1 + D_1)(3A_0^2 + a_0^2 - 6a_0A_0 + \lambda) = 0, \quad (12.11)$$

$$6a_1 + 9a_1^2 - 6a_1A_1 + 3a_1^3 - 6A_1a_1^2 + 3A_1^2a_1 = 0, \quad (12.12)$$

$$6a_1d_1 - 6D_1A_1 + 6a_1^2d_1 - 6a_1(a_1D_1 + A_1d_1)$$
$$+ 6a_1A_1D_1 = 0, \quad (12.13)$$

$$6a_0a_1 - 6A_0a_1 + 6a_0a_1^2 - 6a_1(a_0A_1 + A_0a_1)$$
$$+ 6A_0A_1a_1 = 0, \quad (12.14)$$

$$6a_0a_1d_1 - 6a_1(a_0D_1 + A_0d_1) + 6A_0D_1a_1 = 0, \quad (12.15)$$

$$8ba_1 + 12ba_1^2 - 6bA_1a_1 + 6a_1d_1 + 3a_0^1a_1$$
$$+ 3a_1^2(a_1b + d_1) - 6a_0A_0a_1 - 6a_1A_1(ba_1 + d_1)$$
$$+ 3A_0^2a_1 + A_1^2(ba_1 + d_1) + \lambda a_1 = 0, \quad (12.16)$$

$$6ba_1d_1 - 6ba_1D_1 + (ba_1 + d_1)$$
$$\cdot (3a_1d_1 + 3A_1D_1 - 6a_1D_1 - 6A_1d_1) = 0, \quad (12.17)$$

$$3d_1^2a_1 + 3D_1^2a_1 - 6D_1a_1a_1 = 0, \quad (12.18)$$

$$6ba_0a_1 - 6bA_0a_1 + (a_1b + d_1)$$
$$\cdot (6a_0a_1 + 6A_0A_1 - 6a_0A_1 - 6A_0a_1) = 0, \quad (12.19)$$

$$(3d_1^2 + 3D_1^2 - 6D_1d_1)(ba_1 + d_1) = 0, \quad (12.20)$$

$$(6a_0d_1 + 6A_0D_1 - 6a_0D_1 + A_0d_1)(ba_1 + d_1) = 0, \quad (12.21)$$

$$2b^2a_1 + 3(b^2a_1^2 + d_1^2 + 2ba_1d_1)$$
$$+ (ba_1 + d_1)(3a_0^2 + A_0^2 - 6a_0A_0 + \lambda) = 0, \quad (12.22)$$

for which, we have

$$a_0 = A_0, \, a_1 = A_1 = -2, \, \lambda = -4b, \, d_1 = D_1 = \frac{4b}{3}, (13)$$

$$a_0 = A_0, \, a_1 = A_1 = -2, \, \lambda = 4b, \, d_1 = D_1 = 0, \quad (14)$$

$$a_0 = A_0, \, a_1 = A_1 = -2, \, \lambda = d_1 = D_1 = 0, \quad (15)$$

$$a_0 = a_1 = A_1 = b = 0, \, d_1 = D_1 = -\frac{\lambda}{3}, \quad (16)$$

$$a_0 = a_1 = d_1 = D_1 = 0, \, A_0 \neq 0, \, A_1 = -1,$$
$$\lambda = b - 3A_0^2, \quad (17)$$

$$a_0 = a_1 = A_0 = d_1 = D_1 = 0, \, A_1 = -2, \, \lambda = 4b. \quad (18)$$
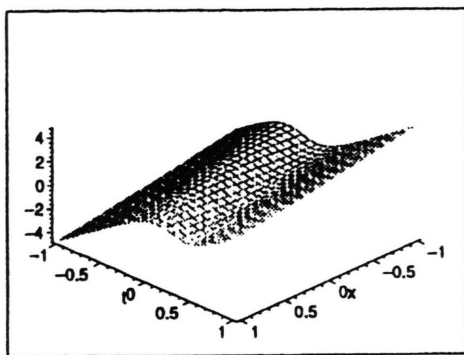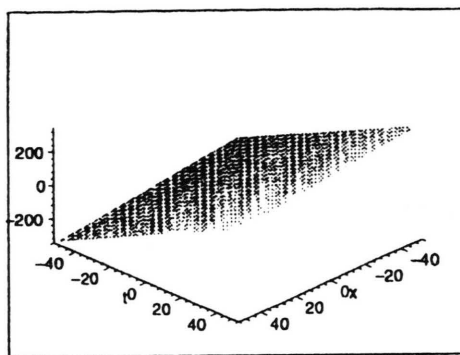
Thus we obtain the solutions of system (1) from (6 - 8), (11) and (13 - 18).
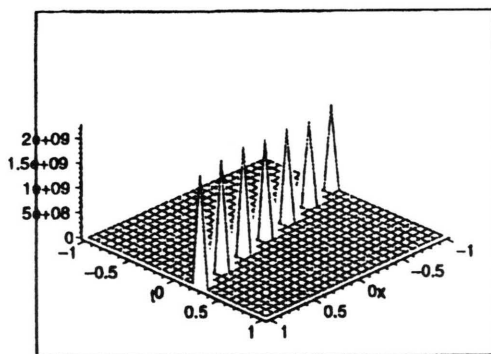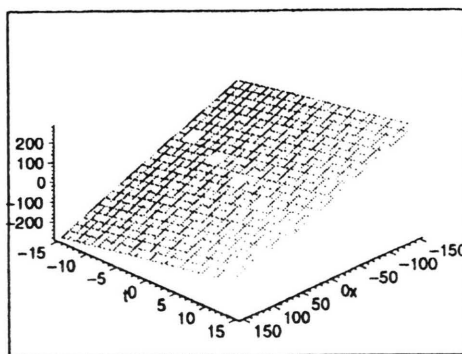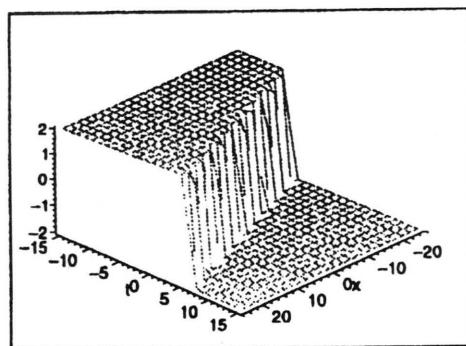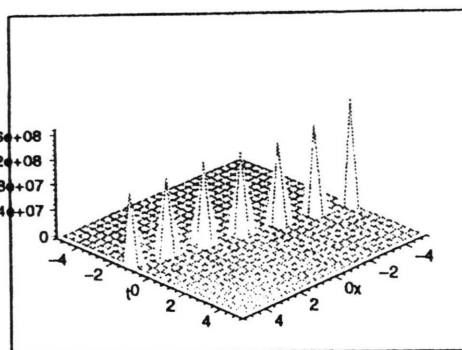
Fig.1a

Fig. 1b

Fig.2a

Fig. 2b

Fig.3

Fig. 4

Case 1: Kink Solitary-like Wave Solution

$$u_1 = v_1 = 2\sqrt{-b}\tanh\sqrt{-b}(x + 4bt + c)$$

$$+ \frac{4b}{3}(x + 4bt + c) + A_0, \ b < 0. \qquad (19)$$

Figure 1a shows a plot of the combined form of the kink soliton solution and the plane $x - 4t = 0$ with $b = -1, c = A_0 = 0$. Figure 1b shows that, when the ranges of x and t become greater and greater, the solution u_1 approximates the plane $-\frac{4}{3}(x - 4t) = 0$. The behaviour of v_1 is similar.

Case 2: Singular Solitary-like Wave Solution

$$u_2 = v_2 = 2\sqrt{-b}\coth\sqrt{-b}(x + 4bt + c)$$

$$+ \frac{4b}{3}(x + 4bt + c) + A_0, \ b < 0. \qquad (20)$$

Figure 2a is a plot of (20) and shows a singular soliton solution $b = -1, c = A_0 = 0$. Figure 2b shows that, when the range of x becomes greater and greater, the solution u_1 approximates a plane with some holes (singular points). The solution develops a singularity at a finite point, i.e., for any fixed $t = t_0$ there always exists an $x = x_0$ at which the solution blows up.
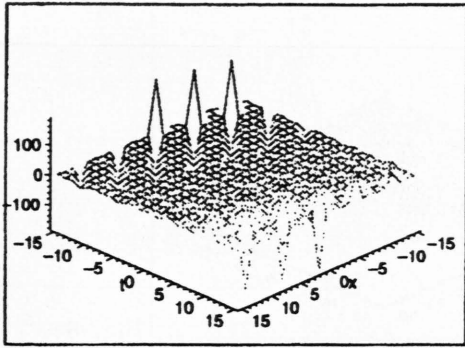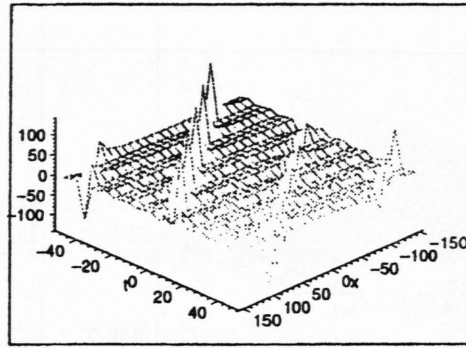
Fig.5

Fig. 6

Case 3: Kink-shaped Solitary Wave Solution

$$u_3 = v_3 = 2\sqrt{-b}\tanh\sqrt{-b}(x - 4bt + c) + A_0, \quad (21)$$

$$b < 0.$$

Figure 3 shows a kink-shaped soliton solution (21) with $b = -1, c = A_0 = 0$.

Case 4: Singular Solitary Wave Solution

$$u_4 = v_4 = 2\sqrt{-b}\coth\sqrt{-b}(x - 4bt + c) + A_0, \quad (22)$$

$$b < 0.$$

Figure 4 shows a singular soliton solution (22) with $b = -1, c = A_0 = 0$.

Case 5: Periodic-like Wave Solution

$$u_5 = v_5 = -2\sqrt{b}\tan\sqrt{b}(x + 4bt + c)$$
$$+ \frac{4b}{3}(x + 4bt + c) + A_0, \ b > 0. \quad (23)$$

Figure 5 shows a plot of a combination of the periodic solution $-2\tan(x + 4t)$ and the plane $\frac{4}{3}(x + 4t) = 0$ with $b = 1, c = A_0 = 0$. It shows that there exist many singular points, i. e. that u_5 is a singular solution. At these singular points the solution blows up.

Case 6: Periodic Wave Solution

$$u_6 = v_6 = -2\sqrt{b}\tan\sqrt{b}(x - 4bt + c) + A_0, \ b > 0. \ (24)$$

Figure 6 shows a plot of the periodic solution (24) with singular points.

Case 7: Rational Fraction Solution

$$u_7 = v_7 = \frac{2}{x} + A_0, \quad (25)$$

Case 8: Rational Solution

$$u_8 = v_8 = -\frac{\lambda}{3}(x - \lambda t + c) + A_0, \quad (26)$$

3. Nine Families of Exact Solutions of System (2)

If we make the transformation

$$u(x, t) = u(\xi), \ v(x, t) = v(\xi), \ \xi = x - \lambda t + c \quad (27)$$

where λ is constant to be determined later and c is an arbitrary constant, then system (2) becomes

$$u''' + 3uu'' - 3vu'' + 3u'^2 - 3u'v' + 3u^2u'$$
$$- 24uvu' + 3v^2u' - 12u^2v' + 6uvv' + \lambda u' = 0,$$

$$v''' - 3uv' + 3vv' - 3u'v' + 3v'^2 + 3u^2v' + 6uvu'$$
$$- 12v^2u' - 24uvv' + 3v^2v' + \lambda v' = 0. \quad (28)$$

If (28) has the form solution

$$u = A_0 + A_1w + B_1\sqrt{1 + \text{sgn}(b)w^2},$$
$$v = a_0 + a_1w + b_1\sqrt{1 + \text{sgn}(b)w^2}, \quad (29)$$

then $w = w(\xi)$ is satisfying (5) and A_0, A_1, B_1, a_0, a_1, b_1 are constants to be determined later.

With the aid of Maple, similar to system (1), substituting (29) into (28) gives rise to a set of equations for $w^i(\sqrt{1 + \text{sgn}(b)w^2})^j (i = 0, 1, 2, 3, 4; j = 0, 1)$. Because they are linearly independent, the coefficients of $w^i(\sqrt{1 + \text{sgn}(b)w^2})^j (i = 0, 1, 2, 3, 4; j = 0, 1)$ should be zero such that we obtain a set of over-determined algebraic polynomials with respect to the unknowns
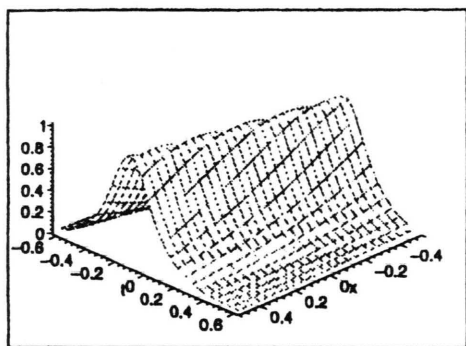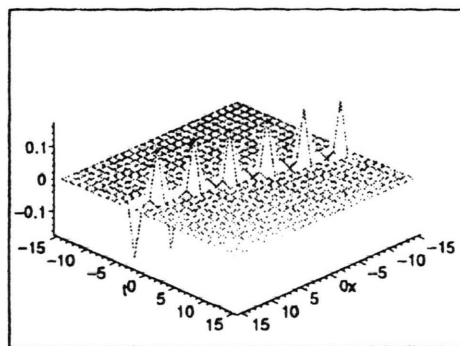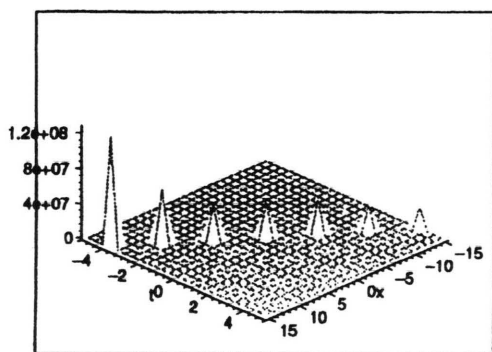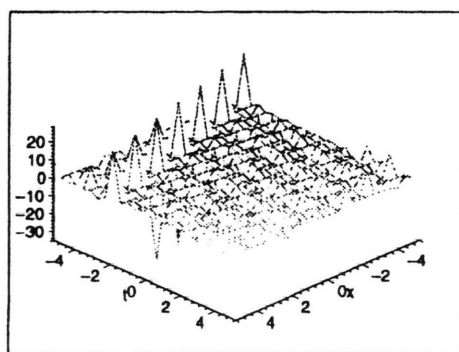
Fig.7

Fig. 8

Fig.9

Fig. 10

λ, b, A_0, A_1, B_1, a_0, a_1, b_1. By solving it and using (29), we can obtain solutions of (2).

Case 1: Kink-shaped Soliton Solutions

$$u_1 = v_1 = \frac{k}{2}\tanh(kx - 2k^3t + c), \qquad (30)$$

The property of (30) is similar to (21).

Case 2: Singular Soliton Solutions

$$u_2 = v_2 = \frac{k}{2}\coth(kx - 2k^3t + c), \qquad (31)$$

The solutions (31) is similar to (22).

Case 3: Bell-shaped Soliton Solutions

$$u_3 = v_3 = \pm\frac{k}{2}i\,\mathrm{sech}(kx+k^3t+c), \ i = \sqrt{-1}. \ (32)$$

The solution is a complex bell-shaped soliton. We give a plot of $|u_3| = \frac{|k|}{2}\mathrm{sech}(kx + k^3t + c)$ in Figure 7.

Case 4: Singular Soliton Solutions

$$u_4 = v_4 = \pm\frac{k}{2}\mathrm{csch}(kx + k^3t + c), \qquad (33)$$

Figure 8 shows a plot of the singular solution (33) with $k = 2, c = 0$.

Case 5: Combined Soliton Solution

$$u_5 = v_5 = \frac{k}{4}[\tanh(kx + k^3t + c) \qquad (34)$$

$$+ i\,\mathrm{sech}(kx + k^3t + c)], \ i = \sqrt{-1}.$$

The solution is complex. It is clear that $|u_5| = \frac{|k|}{4}$.

Case 6: Combined Singular Soliton Solution

$$u_6 = v_6 = \frac{k}{4}[\coth(kx+k^3t+c)+\mathrm{csch}(kx+k^3t+c)], \ (35)$$

Figure 9 shows that the solution (35) is a singular solution.

Case 7: Periodic Solution

$$u_7 = v_7 = -\frac{k}{2}\tan(kx + 2k^3t + c), \qquad (36)$$

Case 8: Periodic Solution

$$u_8 = v_8 = \pm\frac{k}{2}\sec(kx - k^3t + c), \qquad (37)$$

Case 9: Periodic Solution

$$u_9 = v_9 = -\frac{k}{4}[\tan(kx - k^3t + c) + \sec(kx - k^3t + c)],$$
(38)

Figure 10 shows a plot of the periodic solution (38) with $k = 2, c = 0$.

4. Conclusions

In summary, we have obtained eight families of exact solutions of the coupled potential KdV equation and nine families of exact solutions of the modified KdV-type equations by using a Riccati equation, the generalized transformation (9). These solutions contain new kink-shaped soliton solutions, bell-shaped soliton solutions, periodic solutions, rational solutions and singular soliton solutions. The figures of the obtained solutions clearly reflect the properties of the solutions. The singular solutions develop a singularity at a finite point, i. e., for any fixed $t = t_0$, there always exists an $x = x_0$ at which the solution blows up. There is much current interest in the so-called "hot spots" or "blow-up" phenomena [12, 13]. It appears that the singular solutions will model physical phenomena. The method used in this paper has not only some merits in contrast to Fan's method [7, 8] and our previous method [9 - 11], but also generates other new types of exact solutions. With the aid of Maple, the method can be performed by a computer. The approach may be applied to other nonlinear equations in many fields.

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